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Unbiased Estimator of Population Geometric Mean

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Abstract – In this study, attempt has been made on finding unbiased estimator of geometric mean of a population from random sample drawn from it. Description of the estimator along with numerical example has been presented in this article.

Keywords: Population, Geometric Mean, Random Sample, Unbiased Estimator.

1. INTRODUCTION

In the literature of statistical inference, an estimator is defined as a method for calculating parameter's value from sample data [2 , 3 , 15 , 16 , 17 , 18 , 20 , 21] and unbiasedness is regarded as one desirable property/quality of estimator [4 , 5 , 13 , 15 , 21]. Concept of unbiasedness implies that the expected value of an estimator is equal to the true value of the parameter being estimated which further implies that on the average the estimator yields the true/correct value of the parameter to be estimated over sufficient number of trials [14 , 16 , 17 , 19 , 20].

The existing definition of unbiased estimator is based on the concept of mathematical expectation or simply expectation [1 , 2 , 5 , 12 14 , 16 , 19 , 22]. Later on, it was observed that the expectation on the basis of which this unbiased estimator was defined was more specifically the arithmetic expectation [6 , 7]. In addition to the concept of arithmetic expectation, three more concepts of expectation namely geometric expectation [6 , 7 , 8], harmonic expectation [6 , 7 , 9 , 10] and quadratic expectation [11] were introduced/developed in some recent studies. Since the existing definition of expectation is equivalent to the definition of arithmetic expectation [6 , 7], the existing unbiased estimator can also be regarded/termed as arithmetic unbiased estimator.

In many situations, it is required to determine the geometric mean of a population. But the whole population data may not be available for every population. In that case, the common practice is to estimate the population geometric mean from a sample drawn from the population [2 , 3 , 4 , 15 , 18]. An estimator is satisfy some criteria [4 , 5 , 13 , 15 , 21] among which unbiasedness is a desirable one [2 , 16 , 19 , 20].

In this study, attempt has been made on finding unbiased estimator of geometric mean of a population from random sample drawn from it. Description of the estimator has been discussed below along with numerical example.

2. GEOMETRIC UNBIASED IN ESTIMATORS

Let

$$X_1 , X_2 , \ldots\ldots , X_n$$

be a random sample drawn from a population having parameter θ whose true value is θ0

$$\& \qquad T = T(X_1 , X_2 , \ldots\ldots , X_n)$$

be an estimator of the parameter θ.

If

$$E(T) = \theta$$

where $E(T)$ is the expected value of T,

then T is unbiased estimator of parameter θ.

Applying the concept/definition of arithmetic expectation in the general concept/definition of unbiased estimator as mentioned above, it is obtained that

T can be regarded as arithmetic unbiased estimator of parameter θ if

$$E_A(T) = \theta$$

where $E_A(T)$ is the arithmetic expectation of T.

In this connection, it is to be mentioned that this is the concept/definition of unbiasedness usually known/used and are available in the existing literature of statistics.

Now, applying the concept/definition of geometric expectation in the general definition of unbiased estimator, one can obtain the definition of geometric unbiased estimator as follows:

Definition of Geometric Unbiased Estimator

T can be regarded as geometric unbiased estimator of parameter θ if

$$E_G(T) = \theta$$

where $E_G(T)$ is the geometric expectation of T.

Geometric expectation exists and can be defined for strictly positive valued random variable [6 , 7 , 8].

Accordingly, $E_G(T)$ can exist for strictly positive valued estimator T of parameter θ.

This means, geometric unbiased estimator can exist in the case of only strictly positive valued estimator but not for any real valued estimator.

It is to be noted that the value of θ, in this case, is an unknown positive real valued number.

3. UNBIASED ESTIMATOR OF POPULATION GEOMETRIC MEAN

Let a population be consist of the N positive valued observations

$$Y_1 , Y_2 , \dots\dots\dots , Y_N$$

Then the population geometric mean G is given by

$$G = (Y_1 . Y_2 . \dots\dots\dots . Y_N)^{\frac{1}{N}}$$

Suppose

$$y_1 , y_2 , \dots\dots\dots , y_n$$

is a random sample of size n drawn from the population.

Then the sample geometric mean g is given by

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$$g = (y_1 \cdot y_2 \cdot \ldots\ldots\ldots . y_n)^{\frac{1}{n}}$$

Since the sample is random, its each member carries equal probability to assume any observation in the population

This implies,

y_1 assumes the values

$$Y_1 , Y_2 , \ldots\ldots\ldots , Y_N$$

with probabilities

$$P(y_1 = Y_1) = P(y_1 = Y_2) = \ldots\ldots\ldots\ P(y_1 = Y_N) = \frac{1}{N} .$$

Similarly,

y_2 assumes values the values

$$Y_1 , Y_2 , \ldots\ldots\ldots , Y_N$$

with probabilities

$$P(y_2 = Y_1) = P(y_2 = Y_2) = \ldots\ldots\ldots\ P(y_2 = Y_N) = \frac{1}{N} .$$

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y_n assumes values the values

$$Y_1 , Y_2 , \ldots\ldots\ldots , Y_N$$

with probabilities

$$P(y_n = Y_1) = P(y_n = Y_2) = \ldots\ldots\ldots\ P(y_n = Y_N) = \frac{1}{N} .$$

Therefore,

$$E_G(y_1) = Y_1^{1/N} . Y_2^{1/N} . \ldots\ldots\ldots . Y_N^{1/N} = G$$

Similarly,

$$E_G(y_2) = Y_1^{1/N} . Y_2^{1/N} . \ldots\ldots\ldots . Y_N^{1/N} = G$$

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$$E_G(y_n) = Y_1^{1/N} . Y_2^{1/N} . \ldots\ldots\ldots . Y_N^{1/N} = G$$

This implies,

$$E_G(g) = E_G(y_1 \cdot y_2 \cdot \ldots\ldots\ldots \cdot y_n)^{\frac{1}{n}} = \{E_G(y_1 \cdot y_2 \cdot \ldots\ldots\ldots \cdot y_n)\}^{\frac{1}{n}}$$

$$= \{E_G(y_1) \cdot E_G(y_1) \cdot \ldots\ldots\ldots \cdot E_G(y_1)\}^{\frac{1}{n}} = G$$

Hence, g is a geometric unbiased estimator of G.

Thus, the following result is obtained:

"If a random sample is drawn from a population containing positive valued observations, then sample geometric mean is a geometric unbiased estimator of the population geometric mean."

4. NUMERICAL EXAMPLE

Suppose a population consists of the five observed values

3 , 6 , 9 , 12, 15

so that

The Population Geometric Mean = 7.8155132540920556769773007717858

Let us consider random sample of size 2.

There are ${}^5C_2$ = 10 possible random samples of size 2 which are

{3 , 6} , {3 , 9} , {3 , 12} , {3 , 15} , {6 , 9} ,

{6 , 12} , {6 , 15} , {9 , 12} , {9 , 15} , {12 , 15}

The 10 respective Geometric Means of these samples are

4.2426406871192851464050661726291 , 5.1961524227066318805823390245176 , 6.0 ,
6.7082039324993690892275210061939 , 7.3484692283495342945918522241177 ,
8.4852813742385702928101323452582 , 9.4868329805051379959966806332981 ,
10.392304845413263761164678049035 , 11.618950038622250655537796199347 ,
13.416407864998738178455042012388

Now,

Geometric Mean of these 10 sample Geometric Means

= 7.8155132540920556769773007717858

which is the Geometric Mean of the population.

Similarly, if we consider random sample of size 3 then there are ${}^5C_3$ = 10 such possible random samples which are

{3 , 6 , 9} , {3 , 6 , 12} , {3 , 6 , 15} , {3 , 9 , 12} , {3 , 9 , 15} ,

{3 , 12 , 15} , {6 , 9 , 12} , {6 , 9 , 15} , {6 , 12 , 15} , {9 , 12 , 15}

The 10 respective Geometric Means of these samples are

5.451361778496418976673635268981 8 , 6.0 , 6.463304070095651165277880699558 1 ,
6.8682854553199912068482532716381 , 7.3986362229914103044748339694638 ,
8.1432528497847197145542684090385 , 8.6534974218444502939298298646806 ,
9.3216975178615766006329872825672 , 10.259855680060181936118653235263,
11.744602923506590786274808181235

Now,

Geometric Mean of these 10 sample Geometric Means

= 7.8155132540920556769773007717858

which is the Geometric Mean of the population.

Again, if we consider random sample of size 4 then there are $^{5}C_{4}$ = 5 such possible random samples which are

{3 , 6 , 9 , 12} , {3 , 6 , 9 , 15} , {3 , 6 , 12 , 15} , {3 , 9 , 12 , 15} , {6 , 9 , 12 , 15}

The 5 respective Geometric Means of these samples are

6.6400915182019295544527416406 42 , 7.0210419579621478153774229064 5 ,
7.5446005780976124499065930279531 , 8.3494730511412218063225193585634 ,
9.9292527589406193151000153141739

Now,

Geometric Mean of these 5 sample Geometric Means

= 7.8155132540920556769773007717858

which is the Geometric Mean of the population.

In this example thus, sample geometric mean is a geometric unbiased estimator of the population geometric mean.

5. CONCLUSION

It has already been mentioned that unbiasedness is a desirable property of estimator. Accordingly, it is always desired to find such an estimator, of a parameter to be estimated, which is unbiased subject to the fulfilment of the essential criteria of estimator.

In reality, data are not of the same type in every situation. Similarly, parameters to be estimated are not of the same characteristic in the case of different datasets. Accordingly, a specific type of unbiasedness may not be valid and proper for finding unbiased estimator of parameter in the case of every dataset. Thus, the concept of arithmetic unbiasedness may not be or usually is not valid for defining unbiasedness of all types of parameters. Arithmetic unbiasedness is valid for location parameter but not for scale parameter. On the other hand, geometric unbiasedness is valid for scale parameter but not for location parameter. Accordingly, It would be appropriate if it is thought of finding arithmetic unbiased estimator for location parameter and geometric unbiased estimator for scale parameter. It is to be noted mentioned that there is necessity of thinking of more types of unbiasedness due to the necessity of obtaining unbiased estimators of other types of parameters.

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