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Unbiased Estimator of Population Harmonic Mean

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Abstract – The thrust in the study behind this article is to search for an unbiased estimator of population harmonic mean of a population based on random sample drawn from it. Description of the estimator obtained has been presented, in this article, along with numerical example.

Keywords: Population, Harmonic Mean, Random Sample, Unbiased Estimator.

1. INTRODUCTION

An estimator, in statistical literature, is a method for calculating parameter's value to be obtained from data available in sample drawn from the respective population [1 , 2 , 14 - 17 , 19 , 20] while unbiasedness is regarded as a desirable criterion, among some other criteria, of estimator [3 , 4 , 12 , 14 , 20]. Unbiasedness is defined on the philosophy that the expected value of an estimator is equal to the true value of the parameter being estimated i.e. the theoretical average the estimator is the true value of the parameter to be estimated [13 , 15 , 16 , 18 , 19].

Originally, unbiased estimator was defined on the basis of the concept of mathematical expectation [1 , 4 , 11 , 13 , 15 , 18] or equivalently and more specifically arithmetic expectation [5]. In continuation to the definition of arithmetic expectation, three more definitions of expectation had, later on, been introduced which were namely geometric expectation [5 , 6], harmonic expectation [5 , 7 , 8] and quadratic expectation [9]. It is to be noted that the unbiased estimator, defined originally, can also be termed as arithmetic unbiased estimator since it is defined on the basis of arithmetic expectation [5].

In many situations, harmonic mean of a population is required to be known from the data available in sample drawn from the population due to the lack of data on the whole population [1 - 3 , 14 , 17]. The statistical way of obtaining this requirement is to estimate harmonic mean of a population from the data available in the sample drawn from the population with the help of a suitable estimator [1 - 3 , 14 , 17]. A suitable estimator is required to satisfy some criteria [3 , 4 , 12 , 14 , 20] and unbiasedness is a desirable criterion among them [1 , 15 , 18 , 19]. The thrust in the study is to search for an unbiased estimator of population harmonic mean of a population based on random sample drawn from it. Description of the estimator obtained has been presented, in this article, along with numerical example.

2. HARMONIC UNBIASED IN ESTIMATORS

An estimator, which is defined to be a method for evaluating parameter's value, is technically defined as a function of observations in a sample drawn from the respective population [1 , 2 , 14 - 17 , 19 , 20].

Suppose, a random sample of size n drawn from a population having parameter θ is S given by

$$S = \{X_1 , X_2 , \ldots\ldots , X_n\}$$

and

$$T = T(X_1, X_2, \ldots\ldots, X_n)$$

is an estimator of the parameter θ.

The estimator T is regarded as harmonic unbiased estimator of parameter θ if

$$E_H(T) = \theta$$

where $E_H(T)$ is the harmonic expectation of T [5 , 6].

Harmonic unbiased estimator exists in the case of non-zero real valued estimator of non-zero real valued parameter.

3. UNBIASED ESTIMATOR OF POPULATION HARMONIC MEAN

Let a population be consist of the N non-zero real valued observations

$$Y_1, Y_2, \ldots\ldots, Y_N$$

Then the population harmonic mean H is given by

$$H = \frac{1}{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{Y_i}}$$

Suppose,

$$y_1, y_2, \ldots\ldots, y_n$$

is a random sample of size n drawn from the population.

Then the sample harmonic mean h is given by

$$h = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{y_i}}$$

Since the sample is random, its each member carries equal probability to assume any observation in the population

This implies,

y_1 assumes the values

$$Y_1, Y_2, \ldots\ldots, Y_N$$

with probabilities

$$P(y_1 = Y_1) = P(y_1 = Y_2) = \ldots\ldots\; P(y_1 = Y_N) = \frac{1}{N}.$$

This means, y_1 can be any one of the N observations

$$Y_1, Y_2, \ldots\ldots\ldots, Y_N$$

with probability $\frac{1}{N}$.

This implies,

$$E_H(y_1) = \frac{1}{\sum_{i=1}^{N} \frac{1}{N}\frac{1}{Y_i}} = H$$

By the same logic,

$$E_H(y_2) = \frac{1}{\sum_{i=1}^{N} \frac{1}{N}\frac{1}{Y_i}} = H$$

....

$$E_H(y_n) = \frac{1}{\sum_{i=1}^{N} \frac{1}{N}\frac{1}{Y_i}} = H$$

This implies,

$$E_H(h) = E_H\left(\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{y_i}}\right) = \{E_A(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{y_i})\}^{-1}$$

$$= \{\frac{1}{n}\sum_{i=1}^{n} E_A(\frac{1}{y_i})\}^{-1}$$

$$\text{But } E_A(\frac{1}{y_i}) = \{E_H(y_i)\}^{-1} = H^{-1}$$

Therefore,

$$E_H(h) = \{\frac{1}{n}\sum_{i=1}^{n} H^{-1}\}^{-1} = H$$

Hence, h is harmonic unbiased estimator of H.

Thus, the following result is obtained:

"If a random sample is drawn from a population containing non-zero real valued observations then sample harmonic mean is a harmonic unbiased estimator of the population harmonic mean."

4. NUMERICAL EXAMPLE

Let us consider a population P defined by

P = {2 , 4 , 6 , 8, 10}

consisting of five elements/values.

The Population Harmonic Mean H of this population is

H = 4.379562043795620437956204379562

Let us first consider random sample of size 2.

Number of random samples of size 2 possible from this population is $^{5}C_{2}$ = 10.

These 10 possible random samples are

{2 , 4} , {2 , 6} , {2 , 8} , {2 , 10} , {4 , 6} ,
{4 , 8} , {4 , 10} , {6 , 8} , {6 , 10} , {8 , 10}

The 10 respective Harmonic Means of these samples are

2.6666666666666666666666666666667 , 3.0 , 3.2 , 3.3333333333333333333333333333333 ,
4.8000000000000000000000000000001 , 5.3333333333333333333333333333333 ,
5.7142857142857142857142857142857 , 6.8571428571428571428571428571143 ,
7.5000000000000000000000000000002 , 8.8888888888888888888888888888889

Now,

Harmonic Mean of these 10 Sample Harmonic Means

= 4.379562043795620437956204379562

which is the Population Harmonic Mean of the population.

Similarly if we consider random sample of size 3 then the number of random samples of size 3 possible from this population is $^{5}C_{3}$ = 10.

These 10 possible random samples are

{2 , 4 , 6} , {2 , 4 , 8} , {2 , 4 , 10} , {2 , 6 , 8} , {2 , 6 , 10} ,
{2 , 8 , 10} , {4 , 6 , 8} , {4 , 6 , 10} , {4 , 8 , 10} , {6 , 8 , 10}

The 10 respective Harmonic Means of these samples are

3.2727272727272727272727272727272 , 3.4285714285714285714285714285714 ,
3.5294117647058823529411764705883 , 3.7894736842105263157894736842105 ,
3.9130434782608695652173913043478 , 4.1379310344827586206896551724137 ,
5.5384615384615384615384615384614 , 5.8064516129032258064516129032259 ,
6.3157894736842105263157894736843 , 7.6595744680851063829787234042551

Now,

Harmonic Mean of these 10 Sample Harmonic Means

= 4.379562043795620437956204379562

which is the Population Harmonic Mean of the population.

Again if we consider random sample of size 4 then the number of random samples of size 4 possible from this population is ${}^{5}C_{4}$ = 5.

These 5 possible random samples are

$$\{2, 4, 6, 8\}, \{2, 4, 6, 10\}, \{2, 4, 8, 10\}, \{2, 6, 8, 10\}, \{4, 6, 8, 10\}$$

The 5 respective Harmonic Means of these samples are

3.84 , 3.934426229508196721311475409836 , 4.1025641025641025641025641025641 ,
4.4859813084112149532710280373831 , 6.2337662337662337662337662337661

Now,

Harmonic Mean of these 5 Sample Harmonic Means

= 4.379562043795620437956204379562

which is the Population Harmonic Mean of the population.

In this example thus, sample harmonic mean is a harmonic unbiased estimator of the population harmonic mean.

5. CONCLUSION

As per the philosophy of statistical estimation, unbiasedness is a desirable property of estimator [13 , 15 , 16 , 18]. Accordingly, harmonic unbiasedness is also a desirable property of estimator since it is an specific type of unbiasedness. It is to be mentioned that a specific type of unbiasedness may not be valid and proper for finding unbiased estimator of parameter in the case of every dataset and in the case of every parameter. Concept of harmonic unbiasedness can be applicable in the case of dataset containing non-zero real valued observations. It is also to be mentioned that arithmetic unbiasedness is valid for estimating parameter of location type while geometric unbiasedness is valid for estimating parameter of scale type [10]. Since harmonic mean is the reciprocal of arithmetic mean of reciprocals, harmonic unbiasedness is valid for estimating a parameter whose reciprocal is a parameter of location type. Finally, it can be concluded that there is necessity of thinking of more types of unbiasedness due to the necessity of obtaining unbiased estimators of other types of parameters.

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