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Unbiased Estimator of Population Quadratic Mean

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Abstract – Objective of this article is to present an unbiased estimator of population quadratic mean of a population, based on random sample drawn from it, which have been developed in the underlying study. The presentation consists of the theoretical development of the estimator and numerical example of it.

Keywords: Population, Quadratic Mean, Random Sample, Unbiased Estimator.

1. INTRODUCTION

In the literature of statistics, unbiasedness is associated to estimator which is used for obtaining value of parameter from data available in sample drawn from the respective population [1 , 2 , 12 , 13 , 15 - 20]. Quality of estimator is determined by some properties or criteria to be satisfied by it and unbiasedness is regarded as a desirable criterion though not an essential criterion, among some other properties/criteria, of estimator [3 , 14 , 18]. Concept of unbiasedness is based on the philosophy that the expected value of an estimator is to be equal to the true value of the parameter that is to be evaluated which, in other words, is equivalent to the philosophy that the theoretical average the estimator is to be the true value of the parameter to be evaluated [15 - 20].

At the beginning of the development of the theory of unbiased estimator, it was defined on the basis of the concept of mathematical expectation [4 , 21] or more specifically arithmetic expectation [5]. The unbiased estimator, defined then, was simply termed as unbiased estimator [15 - 20] and later on as arithmetic unbiased estimator [8] also since it was based on arithmetic expectation [5 , 7]. In continuation to the development of the theory of unbiased estimator, three more concepts of unbiasedness had, later on, been introduced on the basis of the concepts of geometric expectation [5 , 7], harmonic expectation [5 , 7] & quadratic expectation [6 , 7] and consequently defined which were respectively termed as geometric unbiased estimator [8], harmonic unbiased estimator [8] & quadratic unbiased estimator [8 , 10].

In many situations, quadratic mean [22] of a population is required to be evaluated from the data available in sample drawn from the population. In this case it is required to examine whether the estimator, used for its evaluation, satisfies the criterion of unbiasedness.

Therefore, an attempt has here been made on searching for an unbiased estimator of quadratic mean of a population based on random sample drawn from it. This has been developed by applying the similar methods used in developing geometric unbiased estimator [9], & harmonic unbiased estimator [11]. Description of the development of the estimator obtained has been presented, in this article, along with numerical example.

2. QUADRATIC UNBIASED IN ESTIMATORS

If a random variable X assumes the values

$$x_1 , x_2 , \dots\dots\dots , x_N$$

with respective probabilities

$$p_1, p_2, \ldots\ldots, p_N$$

then the quadratic expectation [6 , 7] of X, denoted by EQ(X), is defined by

$$E_Q(X) = \left(\sum_{i=1}^{N} p_i x_{i_2}\right)^{\frac{1}{2}} \qquad (2.1)$$

while **arithmetic expectation** [5 , 7] **of the random variable, denoted by** EA(X)**, is defined by**

$$E_A(X) = \sum_{i=1}^{N} p_i x_i \qquad (2.2)$$

The two definitions imply that

$$E_Q(X) = \{E_A(X^2)\}^{\frac{1}{2}} \quad \text{or} \quad \{E_Q(X)\}^2 = E_A(X^2) \qquad (2.3)$$

and that

$$E_Q(X^{\frac{1}{2}}) = \{E_A(X)\}^{\frac{1}{2}} \quad \text{or} \quad E_A(X) = \{E_Q(X^{\frac{1}{2}})\}2 \qquad (2.4)$$

Similarly, the quadratic expectation of ψ(X), a function of random variable X, denoted by EQ{ψ(X)}, is defined by $p_i \cdot \{\psi(x_i)\}$

$$E_Q\{\psi(X)\} = \left[\sum_{i=1}^{N} p_i \cdot \{\psi(x_i)\}2\right]^{\frac{1}{2}} \qquad (2.5)$$

while **arithmetic** expectation of ψ(X), a function of random variable X, **denoted by** EA{ψ(X)}**, is defined by**

$$E_A\{\psi(X)\} = \sum_{i=1}^{N} p_i \cdot \{\psi(x_i)\} \qquad (2.6)$$

These two definitions imply that

$$[E_Q\{\psi(X)\}]^2 = E_A[\{\psi(X)\}^2] \qquad (2.7)$$

and that

$$[E_A\{\psi(X)\}]^{\frac{1}{2}} = E_Q\left[\{\psi(X)\}^{\frac{1}{2}}\right] \qquad (2.8)$$

Now let

$$S = \{X_1, X_2, \ldots\ldots, X_n\}$$

be a random sample of size n drawn from a population following a probability distribution having parameter θ

and

$$T = T(X_1, X_2, \ldots\ldots, X_n)$$

be an estimator of θ.

Then T is regarded as quadratic unbiased estimator [8 , 10] of θ if

$$E_Q(T) = \theta \qquad (2.9)$$

and arithmetic unbiased estimator [8] of θ if

$$E_A(T) = \theta \qquad (2.10)$$

Similarly, if

$$S = S(X_1, X_2, \ldots\ldots, X_n)$$

is an estimator of $\phi(\theta)$, a function of parameter θ,

then S is regarded as quadratic unbiased estimator of $\phi(\theta)$ if

$$E_Q(S) = \phi(\theta) \qquad (2.11)$$

and arithmetic unbiased estimator of $\phi(\theta)$ if

$$E_A(S) = \phi(\theta) \qquad (2.12)$$

3. UNBIASED ESTIMATOR OF POPULATION QUADRATIC MEAN

Let a population of the variable Y be consist of the N real valued observations

$$Y_1, Y_2, \ldots\ldots, Y_N$$

So that the quadratic mean [22] of the population of Y denoted by Q(Y), is given by

$$Q(Y) = (\frac{1}{N}\sum_{i=1}^{N} Y_{i_2})^{\frac{1}{2}}$$

and

$$y_1, y_2, \ldots\ldots, y_n$$

are the n elements of a random sample, denoted the sample variable by y, of size n drawn from the population so that the sample quadratic mean, denoted by q(y), is given by

$$q(y) = (\frac{1}{n}\sum_{i=1}^{n} y_{i_2})^{\frac{1}{2}}$$

Since the sample is random, its each member carries equal probability to assume any observation in the population

This implies,

y_1 assumes the values

$$Y_1, Y_2, \ldots\ldots, Y_N$$

with probabilities

$$P(y_1 = Y_1) = P(y_1 = Y_2) = \ldots\ldots\ldots\; P(y_1 = Y_N) = \frac{1}{N}\ .$$

This means, y_1 can be any one of the N elements

$$Y_1\ ,\ Y_2\ ,\ \ldots\ldots\ldots\ ,\ Y_N$$

with probability $\frac{1}{N}$.

This implies,

$$E_Q(y_1)\ =\ \left(\sum_{i=1}^{N}\frac{1}{N}\ Y_{i_2}\right)^{\frac{1}{2}}\ =\ Q(Y)$$

By the same logic,

$$E_Q\,(y_2) = Q(Y)\ ,\ \ldots\ldots\ldots,\ E_Q\,(\,y_n) = Q(Y)$$

This implies,

$$E_Q\,\{q(y)\} = E_Q\,\{(\frac{1}{n}\sum_{i=1}^{n}\ y_{i_2})^{\frac{1}{2}}\} = \{E_A\,(\,\frac{1}{n}\sum_{i=1}^{n}\ y_{i_2}\,)\}^{\frac{1}{2}}$$

$$= \{\,\frac{1}{n}\sum_{i=1}^{n}\ EA\,(\,y_{i2}\,)\}^{\frac{1}{2}}$$

$$\text{But}\quad E_A\,(y_{i}2) = \{E_Q\,(\,y_i)\}2 = \{Q(Y)\}2$$

Therefore,

$$E_Q\,\{q(y)\} = [\,\frac{1}{n}\sum_{i=1}^{n}\ \{Q(Y)\}2]^{\frac{1}{2}}\ = Q(Y)$$

Hence, q(y) is quadratic unbiased estimator of Q(Y).

This leads to the following theorem:

Theorem (3.1): If a random sample is drawn from a population containing real valued observations then sample quadratic mean is a quadratic unbiased estimator of the population quadratic mean.

Now, let us consider the function $\phi(.)$.

Then the population of $\phi(Y)$ consists of the elements

$$\phi(Y_1)\,,\,\phi(Y_2)\ ,\ \ldots\ldots\ldots\ ,\phi(Y_N)$$

while the sample of $\phi(y)$ consists of the elements

$$\phi(y_1)\,,\,\phi(y_2)\ ,\ \ldots\ldots\ldots\ ,\phi(y_n)$$

so that the quadratic mean of the population of ϕ(Y) denoted by Q{ϕ(Y)}, is given by

$$Q\{\phi(Y)\} = [\frac{1}{N}\sum_{i=1}^{N} \{\phi(Y_i)\}_2]^{\frac{1}{2}}$$

and the quadratic mean of the sample of ϕ(y) denoted by q{ϕ(y)}, is given by

$$q\{\phi(y)\} = [\frac{1}{n}\sum_{i=1}^{n} \{\phi(y_i)\}_2]^{\frac{1}{2}}$$

In this case, the possible values of $\phi(y_1)$ are the N values

$$\phi(Y_1), \phi(Y_2), \ldots\ldots\ldots, \phi(Y_N)$$

Also,

$$P\{y_1 = \phi(Y_1)\} = P\{y_1 = \phi(Y_2)\} = \ldots\ldots\ldots = P\{y_1 = \phi(Y_N)\} = \frac{1}{N}$$

This means, $\phi(y_1)$ can be any one of the N values

$$\phi(Y_1), \phi(Y_2), \ldots\ldots\ldots, \phi(Y_N)$$

with probability $\frac{1}{N}$.

This implies,

$$E_Q\{\phi(y_1)\} = [\sum_{i=1}^{N} \frac{1}{N} \{\phi(Y_i)\}_2]^{\frac{1}{2}} = Q\{\phi(Y)\}$$

By the same logic,

$$E_Q\{\phi(y_2)\} = Q(Y), \ldots\ldots\ldots, E_Q\{\phi(y_n)\} = Q\{\phi(Y)\}$$

This implies,

$$E_Q[q\{\phi(y)\}] = E_Q\{(\frac{1}{n}\sum_{i=1}^{n} \phi_i 2)1/2\} \quad \text{where} \quad \phi_i = \phi(y_i)$$

$$= \{E_A(\frac{1}{n}\sum_{i=1}^{n} \phi_i 2)\}1/2$$

$$= \{\frac{1}{n}\sum_{i=1}^{n} EA(\phi_i 2)\}1/2$$

$$\text{But} \quad E_A(\phi_i 2) = \{E_Q(\phi_i)\}2 = [Q\{\phi(Y)\}]2$$

Therefore,

$$E_Q[q\{\phi(y)\}] = Q\{\phi(Y)\}$$

Hence, q(y) is quadratic unbiased estimator of Q(Y).

This leads to the following theorem:

Theorem (3.2): If a random sample is drawn from a population containing real valued observations then sample quadratic mean of a function of the elements in the sample is a quadratic unbiased estimator of the population quadratic mean of the function of the elements in the population.

Special Case: The sample quadratic means of

$$y_2,\ y_3,\ y\,4,\ y^{\frac{1}{2}},\ y^{\frac{1}{3}},\ y^{\frac{1}{2}},\ \ldots\ldots,\ y^p,\ y^{\frac{1}{p}}$$

are quadratic unbiased estimators of the corresponding population quadratic means of

$$Y_2,\ Y_3,\ Y_4,\ y^{\frac{1}{2}},\ Y^{\frac{1}{3}},\ Y^{\frac{1}{2}},\ \ldots\ldots,\ Y^p,\ Y^{\frac{1}{p}}$$

respectively.

4. NUMERICAL EXAMPLE

Let us consider the example, in an earlier study [11], of population P where

$$P = \{2, 3, 5, 7, 11\}$$

consisting of five elements so that

Population Quadratic Mean = 6.449806198638839721893290584243

If we consider random sample of size 2 then the number of such possible random samples to be drawn from this population is ${}^{5}C_{2} = 10$

and the 10 Sample Quadratic Means of these 10 samples are:

2.5495097567963924150141120545114 , 3.8078865529319541428307055135792 ,
5.1478150704935001578986847320988 , 7.9056941504209483299972338610818 ,
4.1231056256176605498214098559741 , 5.3851648071345040312507104915403 ,
8.0622577482985496523666132303038 , 6.0827625302982196889996842452021 ,
8.5440037453175311678716483262397 , 9.2195444572928873100022742817628

Now,

Quadratic Mean of these 10 Sample Quadratic Means

= 6.449806198638839721893290584243

= Population Quadratic Mean

Similarly the number of possible random samples, of size 3, to be drawn from this population is ${}^{5}C_{3} = 10$

and the 10 Sample Quadratic Means of the 10 samples are:

3.5590260840104370702705079885319 , 4.5460605656619519642528658506432 ,
6.6833125519211404502529128341505 , 5.0990195135927848300282241090228 ,
7.0710678118654752440084436210485 , 7.6157731058639082856614110271583 ,
5.2599112793531666817627647633141 , 7.1879528842826082249113139330004 ,
7.7244201508376450215466993651165 , 8.0622577482985496523666132303038

Now,

Quadratic Mean of these 10 Sample Quadratic Means

= 6.4498061986388397218932905842 43

= Population Quadratic Mean

Again, if we consider random sample of size 4 then the number of such possible random samples to be drawn from this population is ${}^{5}C_{4}$ = 5

and the 5 Sample Quadratic Means of these 5 samples are:

4.6636895265444075227772377711603 , 6.3047601064592457656143129172666 ,
6.7638746292343414487129374086906 , 7.0533679898329422126160818454387 ,
7.1414284285428499979993998113673

Now,

Quadratic Mean of these 5 Sample Quadratic Means

= 6.449806198638839721893290584243

= Population Quadratic Mean

Thus, this is an example which shows that the sample quadratic mean is a quadratic unbiased estimator of population quadratic mean.

5. CONCLUSION

Quadratic unbiasedness can be regarded as a desirable, though not essential, property/criterion of estimator since it is a specific unbiasedness of estimator and since unbiasedness, in general, is regarded as a property/criterion of estimator which, though not essential, is desirable. It is to be mentioned that a specific type of unbiasedness may not be valid and proper for finding unbiased estimator of parameter in the case of every dataset and in the case of every parameter. Concept of harmonic unbiasedness can be applicable in the case of dataset containing non-zero real valued observations.

It is also to be mentioned that arithmetic unbiasedness is valid for estimating parameter of location type while geometric unbiasedness is valid for estimating parameter of scale type and harmonic unbiasedness is valid for estimating parameter whose reciprocal is a parameter of location type [19]. Since quadratic mean [22] is the square root of arithmetic mean of squares [23], quadratic unbiasedness is valid for estimating a parameter whose square is a parameter of location type. At this stage it is to be mentioned that similar logic and similar method can probably be applied in developing more concepts of unbiasedness due to the necessity of obtaining unbiased estimators of parameters of different types.

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